The yielding of the Earth to disturbing forces. By A. E. H. Love, D.Sc., F.R.S., Sedleian Professor.

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The Earth, not being an absolutely rigid body, must be deformed by the tide-generating forces of the Sun and Moon. The object of the first part of the paper is to determine the amount by which the surface actually yields to these forces, or, what is the same thing, the height of the corporeal tide raised by them on the Earth.

It is assumed that, for a first approximation, the Earth may be regarded as spherical, and the surfaces of equal density within it as spheres concentric with the boundary. It is assumed that, for a second approximation, the boundary may be regarded as an oblate ellipsoid of revolution, and the surfaces of equal density also as oblate ellipsoids of revolution, the ellipticities of all these surfaces being subject to the law of "fluid equilibrium." This is the assumption customarily made in the theory of the Figure of the Earth; it does not involve the further assumption of any special law of density, *e.g.* that of Laplace. It is assumed also that, when small disturbing forces, such as the tide-generating forces, act upon the Earth, the deformation produced in it obeys a linear law of elasticity. With these limitations the substance may be compressible or incompressible, it may be possessed of a finite degree of rigidity or be altogether devoid of that quality, and there is an infinite number of ways in which its mass may be distributed.

The effects produced by tide-generating forces in such a body are a deformation of the surface and a state of strain within the body, but, whatever the constitution of the body may be, subject to the limitations stated above, this state of strain and superficial deformation are connected with the forces in a certain way. Let W_2 denote the potential of the tide-generating forces, expressed as a spherical solid harmonic of the second degree. Then the potential of the disturbed body at any point differs from that of the undisturbed body by the addition of a term of the form $K(r)W_2$, where $K(r)$ denotes some function of the distance r of the point from the centre. The value of $K(r)$ at the surface is a certain constant number k, to be determined, if possible, by observation. The deformation of the surface of the body, or the height of the corporeal tide raised on it, is expressible in the form hW_2/g, where g denotes the value of gravity at the surface, and h is a certain number, to be determined, if possible, by observation. When h is known, the amount by which the surface yields to tide-generating forces is known.

The ordinary uncorrected equilibrium theory of the oceanic tides on a rigid body of the same size and mass as the Earth makes the height of the tide equal to W_2/g. Since the potential of the tide-

generating forces near the surface of a yielding body is increased by the deformation in the ratio $1+k:1$, the corresponding expression for the deformation of the surface of an ocean resting on the yielding body, and executing tidal motions according to the equilibrium law, would be $(1+k)W_2/g$. But since the surface of the body is deformed by an amount hW_2/g, the observable height of oceanic tide, or rise and fall of the ocean relative to the body, is $(1+k-h)W_2/g$. In the case of the Earth this expression needs to be corrected for the depth of the ocean, for its self-attraction, and for the existence of continents, and it should then give correctly the height of the tides of long period, such as the fortnightly tide. Apart from such corrections, the heights of these tides are less than the corresponding "true equilibrium heights" in the ratio $1+k-h:1$.

A similar result is obtained in connection with the behaviour of a horizontal pendulum. When sufficient care is taken to eliminate the effects of air-currents, changes of temperature, and the presence of the observer, the observed displacement of the pendulum by the tide-generating force of the Sun and Moon is $1+k-h$ of what it would be if the Earth were absolutely rigid.

The analysis of tidal observations made by Sir G. Darwin for the second edition of Thomson and Tait's *Natural Philosophy*, a more recent reduction of more numerous observations by W. Schweydar,* certain experiments with horizontal pendulums made by Ehlert and v. Rebeur Paschwitz †, and, above all, the recent remarkable investigations of O. Hecker ‡ on the displacement of horizontal pendulums mounted in an underground chamber, concur in leading to the result that the number $1+k-h$ is very nearly $\frac{2}{3}$. One outcome of all this observational work is to establish the approximate equation

$$h-k=\tfrac{1}{3}.$$

The periodic variation of latitude discovered by S. C. Chandler § gives the means of obtaining a second equation by which the numbers h and k may be determined. It has long been known that if the Earth were absolutely rigid, the period of the variations of latitude, if any, would be about 306 days. When Chandler found that the actual period is about 427 days, Newcomb ‖ pointed out that the lengthening of the period must be due to a deformation of the Earth, the figure being adjusted at any time to the rotation about the instantaneous axis. The adjustment of the figure involves a deformation of the surface and an internal state of strain, leading to an inequality in the potential of the Earth; and the amounts of the deformation and the inequality of potential are the same as they would be if the Earth were subject to a

* *Beiträge zur Geophysik*, vol. ix. (1907), p. 41.

† Reduced by Schweydar, *loc. cit.*

‡ *Veröffentlichung d. k. Preusz. geodätischen Institutes*, No. 22 (1907).

§ *Astronomical Journal*, vol. xi. (1891).

‖ *Monthly Notices*, 1892.

certain system of forces. These forces have, in common with the tide-generating forces, the property of being derived from a potential which is a spherical solid harmonic of the second degree, and the effects may therefore be specified by the same two numbers h and k by which the corporeal tides are specified. A theoretical investigation of the period of variations of latitude in the case of a yielding body was made by S. S. Hough,* on the supposition that the body might be treated as homogeneous and absolutely incompressible. G. Herglotz † has shown how to dispense with the assumption of homogeneity. A revision of Herglotz' work shows that it is possible to dispense with the assumption of absolute incompressibility also, and it appears that the period depends upon the number k and not upon the number h.

The equation found for k is

$$k\frac{a\omega^2}{2g}=\left(\epsilon-\frac{a\omega^2}{2g}\right)\left(1-\frac{\tau_0}{\tau}\right),$$

in which a denotes the mean radius, ω the angular velocity of rotation, g the value of gravity at the surface, ϵ the ellipticity of the meridians, τ the actual period, τ_0 the corresponding period which would exist if the Earth were absolutely rigid. All these quantities are known with sufficient approximation, and it appears that

$$k=\tfrac{4}{15}\text{ nearly.}$$

This result is obtained without using any additional assumption beyond those stated at the beginning of this Abstract, and Larmor ‡ has shown that, of these, the assumption that the ellipticities of the surfaces of equal density are subject to the law of fluid equilibrium may be discarded. The result is therefore established for almost any conceivable constitution of the matter within the Earth.

From the two approximate equations

$$h-k=\tfrac{1}{3}\quad\text{and}\quad k=\tfrac{4}{15}$$

it follows that

$$h=\tfrac{3}{5}\text{ nearly,}$$

or the height of the corporeal tide raised on the Earth is about three-fifths of the "true equilibrium height" of the corresponding oceanic tide.

The remainder of the paper is occupied with discussions of the rigidity which must be assigned to the Earth in order that the equations $h=\frac{3}{5}$ and $k=\frac{4}{15}$ may be verified approximately. For this purpose the substance of the Earth is treated as absolutely incompressible. Lord Kelvin § in his estimate of the rigidity used an equation equivalent to $h-k=\frac{1}{3}$, combining it with the hypotheses of homogeneity and absolute incompressibility, from which

* *Phil. Trans.* (A), vol. clxxxvii. (1896), p. 319.
† *Zeitschr. f. Math. u. Phys.*, vol. lii. (1905), p. 275.
‡ *Proc. R. Soc.*, London, vol. lxxxii. (1909), p. 89.
§ Thomson and Tait, *Nat. Phil.*, part ii., §§ 735–737 and 838–847.

hypotheses it would follow that $k=\frac{3}{5}h$ and $h=\frac{5}{6}$. This method yields rigidity about equal to that of steel ($7{\cdot}6\times10^{11}$ dynes per square cm.). If h is taken to be $\frac{3}{5}$ and the substance is assumed to be homogeneous and absolutely incompressible, the rigidity comes out as $1{\cdot}2\times10^{12}$ dynes per square cm. If k is taken to be $\frac{4}{15}$ and the substance is assumed to be homogeneous and absolutely incompressible, the rigidity comes out as $1{\cdot}7\times10^{12}$ dynes per square cm. The first method makes h right and k wrong, the second makes k right and h wrong, and it appears to be impossible to make both these numbers have the right values if the substance, assumed absolutely incompressible, is also taken to be homogeneous. The two methods, however, agree in making the rigidity much greater than it was estimated to be by Lord Kelvin.

If the Earth is taken to consist of a homogeneous nucleus of density 8·206 enclosed in a homogeneous shell of density 3·2, the ratio of the radius of the nucleus to the outer radius of the shell being 0·78 (Wiechert's law), and if the nucleus and the shell are assumed to be of the same rigidity, Herglotz found that, to make the Chandler period have the right value, 427 days, the rigidity must be about $11{\cdot}68\times10^{11}$ dynes per square cm. Schweydar found that to make the displacement of a horizontal pendulum by tide-generating forces equal to two-thirds of what it would be if the Earth were absolutely rigid, the rigidity must be about $6{\cdot}3\times10^{11}$ dynes per square cm. He also found that both conditions can be satisfied if the nucleus has rigidity $2{\cdot}02\times10^{12}$ and the shell has rigidity $0{\cdot}9\times10^{11}$ dynes per square cm. These results show that condensation of mass towards the centre and increase of rigidity in the central portions of the Earth may compensate for a very considerable defect of rigidity in the superficial portions.

This result, however, cannot be pushed so far as to support the hypothesis that the nucleus is separated from the shell by a layer of matter devoid of rigidity, or that there is such a layer within the shell. To illustrate this statement the problem of tidal distortion of a solid nucleus and solid shell separated by a fluid layer is worked out in the paper, and it is found that, even if the nucleus were absolutely rigid, the shell 1400 km. thick, and the fluid layer thin, the rigidity which the shell would need to have, in order that its surface might yield no more to tidal disturbing forces than the surface of the Earth does, would be about five times as great as the rigidity of steel. With a yielding nucleus, a thicker layer of fluid, or a thinner shell, the requisite rigidity would be still greater. It is concluded that the Earth cannot consist of a fluid body enclosed in a solid crust, or of a solid nucleus separated from a solid enclosing shell by a fluid layer. This conclusion would not negative the possible existence of areas of continental dimensions beneath which there might be molten matter, but it would mean that such areas must be isolated; the molten matter could not form a continuous layer separating the inner portions of the Earth's body from the outer portions.

The Relation of the Earth's Free Precessional Nutation to its Resistance against Tidal Deformation. By Prof. J. Larmor, Sec. R.S.

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The modern investigation of the wandering of the Earth's axis of rotation, considered as a physical problem relating to the actual non-rigid Earth, may be said to have been initiated in Lord Kelvin's address to the Physical Section of the British Association in 1876. After referring* to the scrutiny of the recorded observations of change of latitudes, conducted by Peters in 1841 and independently by Maxwell in 1851, in search of the regular Eulerian free period of 306 days which would belong to a rigid Earth, with negative results, he insisted that the irregular motions brought out in these analyses are not merely due to instrumental imperfections, but represent true motions of the Pole, due to displacement of terrestrial material. For example, he estimates that existing shifts of material, of meteorological type, are competent to produce displacements of the axis of rotation ranging from 1/2 to 1/20 of a second of arc. A sudden shift of material on the Earth will not at once affect the axis of rotation, but will start it into motion round the altered axis of inertia, with a period of 306 days if the Earth were rigid, which will go on displacing the Pole until it is damped out by the frictional effects of the tidal motions thus originated. A radius of rotation of 1 second of arc would raise an ocean tide of the same period as the rotation, having as much as 11 cm. of maximum rise and fall. Thus the motion of the Pole is to be considered as continually renewed by meteorological and other displacements, as it is damped off by tidal and elastic friction; it was therefore, perhaps, not to be expected that it would show much periodicity, though the movements were eminently worthy of close investigation. Their nature was examined more closely by Newcomb at Kelvin's request; but not much more had been done regarding their cause when Chandler announced that the records of changes of latitude did actually indicate a period of precession—of 427 days, however, instead of the Eulerian period of 306 days, which, if any, had previously been taken for granted. Soon after, in 1890, observations were organised systematically by the International Geodetic Union on the motion of Prof. Foerster, of Berlin; and already, in 1891, he was able to inform Lord Kelvin that a comparison of European observations with synchronous ones made at Honolulu gave direct proof of his conclusion of 1876 (*supra*), "that irregular movements of the Earth's axis to the extent of half a second may be produced by the temporary changes of sea level due to meteorological causes."†

* Reprint in *Popular Lectures and Addresses*, vol ii., see pp. 262–272.

† "Presidential Address R.S.," Nov. 30, 1892; *Popular Lectures* ...., vol. ii. p. 504. Lord Kelvin's investigations up to 1876 are collected in *Math. and Phys. Papers*, vol. iii.; especially pp. 312–350.